MATH 161: Practice Final

Name: ____

Directions: No technology, internet, or notes. **Simplify all expressions + fully show all work for full credit**. If you have a question, ask me. Good luck!

Problem	Score	Points
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
		100

1. Find the following limits:

(a) $\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} = \frac{0}{0}$

$$\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} \overset{L'H}{=} \lim_{h\to 0} \frac{\frac{1}{2}(1+h)^{-\frac{1}{2}}}{1} = \lim_{h\to 0} \frac{1}{2\sqrt{1+h}} = \lim_{h\to 0} \frac{1}{2\sqrt{1}}$$

$$= \boxed{\frac{1}{2}}$$

(b) $\lim_{x\to 0} \frac{1}{x} - \frac{1}{x^2-x} = \infty - \infty$

change to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form then use L'H

$$\lim_{x\to 0} \frac{1}{x} - \frac{1}{x(x-1)} = \lim_{x\to 0} \frac{x-1}{x(x-1)} - \frac{1}{x(x-1)} = \lim_{x\to 0} \frac{x-2}{x(x-1)}$$

$$= \frac{-2}{0(0-1)}$$

$$= \frac{-2}{0} \leftarrow \boxed{DNE}$$

(c) $\lim_{x\to 1} 2x^{100} + 4x^{55} - 8x^3 + 10x^{-2}$

$$= 2\cdot 1^{100} + 4\cdot 1^{55} - 8\cdot 1^3 + \frac{10}{1^2}$$

$$= 2 + 4 - 8 + 10 = \boxed{8}$$

(d) $\lim_{x\to 1} \frac{x^2}{1-\cos x} = \frac{0^2}{1-\cos(0)} = \frac{0}{0} \leftarrow$ use L'H

this was supposed to be $x \to 0$.

$$\lim_{x\to 0} \frac{x^2}{1-\cos x} \overset{L'H}{=} \lim_{x\to 0} \frac{2x}{\sin(x)} \overset{L'H}{=} \lim_{x\to 0} \frac{2}{\cos(x)} = \frac{2}{\cos(0)} = \frac{2}{1} = \boxed{2}$$

$$\frac{2\cdot 0}{\sin(0)} = \frac{0}{0} \checkmark$$

2. Find all of the local maximums and minimums of

$$f(x) = x + \frac{16}{x}$$

We will use the First Derivative Test.

① Find locations of local maximum/minimums.

$$f(x) = x + 16x^{-1} \longrightarrow f'(x) = 1 - 16x^{-2} = 1 - \frac{16}{x^2}$$

$$= \frac{x^2}{x^2} - \frac{16}{x^2} = \frac{x^2 - 16}{x^2}$$

$\longleftarrow$ $f'(x)$ DNE at $x = 0$

Solve $f'(x) = 0$. $x^2 \cdot \frac{x^2 - 16}{x^2} = 0 \cdot x^2 \longrightarrow x^2 - 16 = 0$

$$\longrightarrow x^2 = 16 \longrightarrow x = \pm\sqrt{16} = \pm 4$$

② Use $f'(x)$ to determine if those locations have a local min or max.

f': $+$ | -4 | $-$ | 0 | $-$ | 4 | $+$

local max of $f(-4) = -8$ at $x = -4$,

local min of $f(4) = 4 + 4 = 8$ at $x = 4$

$$f'(x) = \frac{x^2 - 16}{x^2} = \frac{(x-4)(x+4)}{x^2}$$

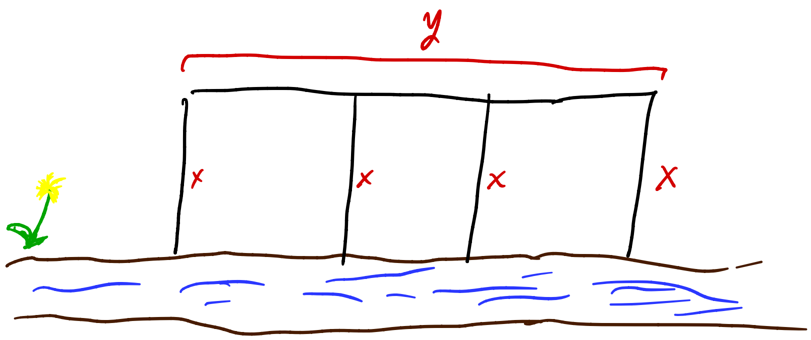
$f'(-5) = \frac{- \cdot -}{+} = +$ $\qquad$ $f'(1) = \frac{- \cdot +}{+} = -$

$f'(-1) = \frac{- \cdot +}{+} = -$ $\qquad$ $f'(5) = \frac{+ \cdot +}{+} = +$

3. Suppose someone owns 4000 meters of fencing. They wish to create a rectangular piece of grazing land where one side is along a river. This means no fence is needed for the side along the river. Moreover, they wish to subdivide the rectangle into three sections with two pieces of fence, both of which are parallel to the sides not along the river.

   (a) What are the dimensions of the largest area that can be enclosed?

①



② maximize $A = x \cdot y$ ← middle two pieces of fence do not change the enclosed area.

③ eliminate y: $4x + y = 4000$

$$y = 4000 - 4x \quad \text{so}$$

$$A = x \cdot y = \underbrace{x(4000 - 4x)}_{f(x)} \quad [0, 1000]$$

Use Closed Interval method:

$$f(x) = x(4000 - 4x) = 4000x - 4x^2$$

$$f'(x) = 4000 - 8x$$

$$4000 - 8x = 0 \longrightarrow x = 500$$

Now $f(500) = 500 \cdot (4000 - 2000)$

$$= \boxed{500 \cdot 2000}$$

$$f(0) = 0 \cdot 4000 = 0$$

$$f(1000) = 1000 \cdot 0 = 0$$

So the dimensions are

$x = 500$ meters, $y = 2000$ meters

4. Find the absolute minimum and maximum value, if any, of

$$f(x) = \frac{1}{8}x^2 - 4\sqrt{x} \qquad [0, 9]$$

We can use the Closed Interval Method.

① $f(x) = \frac{1}{8}x^2 - 4x^{\frac{1}{2}}$

$f'(x) = \frac{1}{4}x - 2x^{-\frac{1}{2}}$

$= \frac{1}{4}x - \frac{2}{\sqrt{x}}$

$= \frac{x}{4} \cdot \frac{\sqrt{x}}{\sqrt{x}} - \frac{2}{\sqrt{x}} \cdot \frac{4}{4}$

$= \frac{x^{\frac{3}{2}}}{4\sqrt{x}} - \frac{8}{4\sqrt{x}}$

$= \frac{x^{\frac{3}{2}} - 8}{4\sqrt{x}}$ ← DNE at $x = 0$

Solve $f'(x) = 0$

$4\sqrt{x} \cdot \frac{x^{\frac{3}{2}} - 8}{4\sqrt{x}} = 0 \cdot 4\sqrt{x}$

$x^{\frac{3}{2}} - 8 = 0$

$\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} = (8)^{\frac{2}{3}}$

$x = 8^{\frac{2}{3}} = \sqrt[3]{8^2}$

$= \sqrt[3]{(2^3)^2}$

$= \sqrt[3]{2^6} = 4$

② $f(4) = \frac{1}{8} \cdot 4^2 - 4\sqrt{4}$

$= \frac{16}{8} - 4 \cdot 2 = 2 - 8 = -6$ ← abs min of -6 at $x = 4$

$f(0) = \frac{1}{8}0^2 - 4\sqrt{0} = 0$ ← abs max of 0 at $x = 0$

$f(9) = \frac{1}{8}9^2 - 4\sqrt{9}$

$= \frac{81}{8} - 4 \cdot 3$

$= \frac{81}{8} - \frac{96}{8} = -\frac{15}{8} = -1\frac{7}{8}$

5. Using implicit differentiation, find dy/dx of

$$x^2 \cos^3(y) = 5 + 2xy^2$$

product rule

$$\frac{d}{dx}\left[x^2 \cos^3(y)\right] = \frac{d}{dx}\left[5 + 2xy^2\right]$$

triple chain rule

product rule

$$\cos^3(y)\frac{d}{dx}[x^2] + x^2\frac{d}{dx}[\cos^3(y)] = \frac{d}{dx}[5] + 2\frac{d}{dx}[xy^2]$$

$$\cos^3(y)\cdot 2x + x^2\, 3\cos^2(y)\cdot\frac{d}{dx}[\cos(y)] = 0 + 2\left(y^2\cdot 1 + x\cdot 2y\cdot\frac{dy}{dx}\right)$$

$$2x\cos^3(y) + 3x^2\cos^2(y)\cdot(-\sin(y))\cdot\frac{dy}{dx} = 2y^2 + 4xy\frac{dy}{dx}$$

$$-4xy\frac{dy}{dx} - 3x^2\cos^2(y)\sin(y)\frac{dy}{dx} = 2y^2 - 2x\cos^3(y)$$

$$\frac{dy}{dx}\left(-4xy - 3x^2\cos^2(y)\sin(y)\right) = 2y^2 - 2x\cos^3(y)$$

$$\boxed{\frac{dy}{dx} = \frac{2y^2 - 2x\cos^3(y)}{-4xy - 3x^2\cos^2(y)\sin(y)}}$$

6. The number of housing starts is related to the mortgage rate by the equation

$$11N^3 + r = 16$$

where $N(t)$ is the number of housing starts (in units of a million) and $r(t)$ is the mortgage rate (in percent per year). What is the rate of change of the mortgage rate with respect to time when the number of housing starts is 1 million and is decreasing by $1/33$ of a million per year?

related rates problem

① N – number of housing starts (million)

r – mortgage rate (% per year)

② $N = 1$, $\frac{dN}{dt} = -\frac{1}{33}$, want: $\frac{dr}{dt}$

③ $11N^3 + r = 16$

④ $\frac{d}{dt}\left[11N^3 + r\right] = \frac{d}{dt} 16$

$$11 \cdot 3N^2 \cdot \frac{dN}{dt} + \frac{dr}{dt} = 0$$

$$\frac{dr}{dt} = -33 \cdot N^2 \cdot \frac{dN}{dt}$$

⑥ $\frac{dr}{dt} = -33 \cdot 1^2 \cdot \left(-\frac{1}{33}\right) =$ 1 % per year

7. Find the derivative of the following:

(a) $f(x) = \left(x^2 + \boxed{\frac{1}{e^{-x}}}\right)^{3/2}$ $\longrightarrow$ $\frac{1}{e^{-x}} = \frac{1}{\frac{1}{e^x}} = e^x$

$$f'(x) = \frac{3}{2}\left(x^2 + e^x\right)^{\frac{1}{2}} \cdot \frac{d}{dx}\left[x^2 + e^x\right]$$

$$= \frac{3}{2}\left(x^2 + e^x\right)^{\frac{1}{2}} \cdot \left(2x + e^x\right)$$

(b) $f(x) = (\ln(x))^3$

$$f'(x) = 3(\ln(x))^2 \cdot \frac{d}{dx}\ln(x)$$

$$= 3(\ln(x))^2 \cdot \frac{1}{x} = \frac{3(\ln(x))^2}{x}$$

(c) $f(x) = x^{\sin x}$

$$y = x^{\sin x}$$

$$\ln y = \ln x^{\sin(x)} = \sin(x) \cdot \ln(x)$$

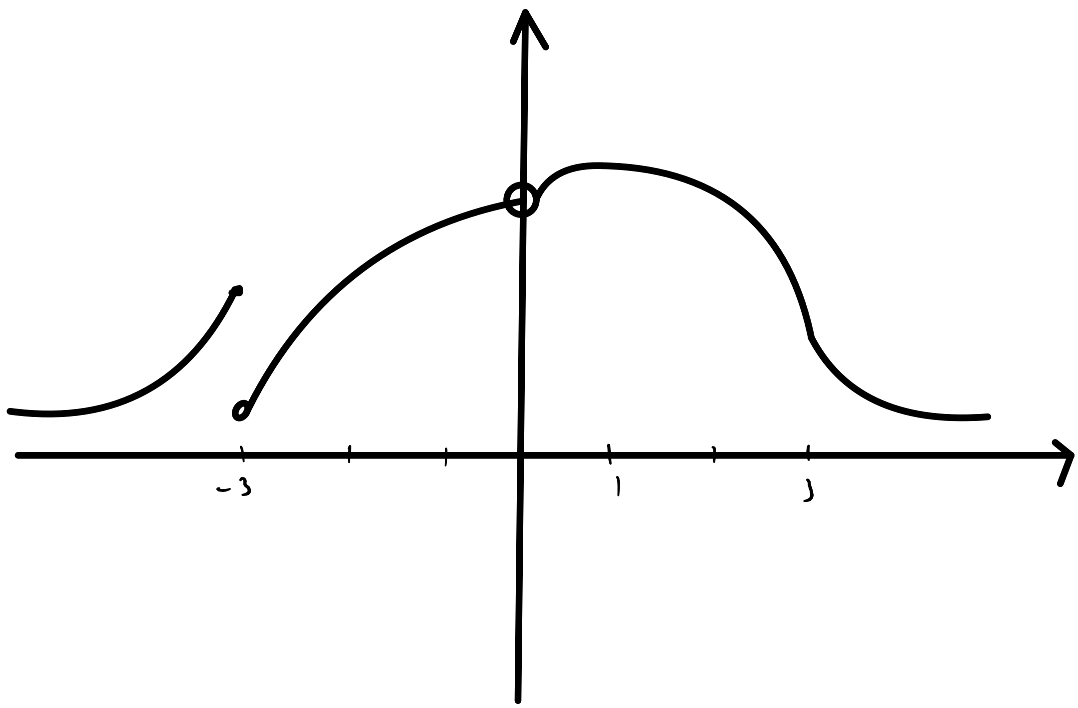
$$\frac{d}{dx}\ln y = \frac{d}{dx}\left[\sin(x)\ln(x)\right]$$

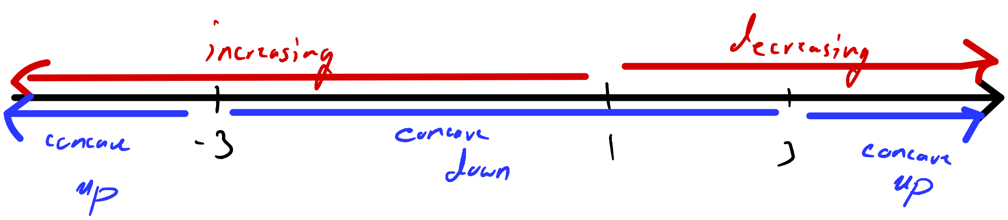
$$\frac{1}{y} \cdot y' = \ln(x) \cdot \cos(x) + \sin(x) \cdot \frac{1}{x}$$

$$y' = y\left(\ln(x)\cos(x) + \frac{\sin(x)}{x}\right) = x^{\sin(x)}\left(\ln(x)\cos(x) + \frac{\sin(x)}{x}\right)$$

8. Sketch a graph of a function satisfying:

   (a) $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$
   (b) $f''(x) > 0$ on $(-\infty, -3) \cup (3, \infty)$, $f''(x) < 0$ on $(-3, 3)$
   (c) $\lim_{x \to -3} f(x)$ does not exist
   (d) $f(-3) = 4$
   (e) $x = 0$ is a critical number





9. **Use the limit definition of the derivative** to find the derivative of $f(x) = \sqrt{x}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$= \boxed{\frac{1}{2\sqrt{x}}}$$

10. Find the intervals of concavity of

$$f(x) = \frac{x+1}{x-1}$$

$$f'(x) = \frac{(x-1)\cdot\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x-1)}{(x-1)^2}$$

$$= \frac{x-1 - x-1}{(x-1)^2} = \frac{-2}{(x-1)^2} = -2(x-1)^{-2}$$

to avoid quotient rule

$$f''(x) = -2(-2)(x-1)^{-3}\cdot\frac{d}{dx}[x-1]$$

$$= 4(x-1)^{-3} = \frac{4}{(x-1)^3}$$

DNE at $x=1$.

can't solve $f''(x)=0$ since

$$\cancel{(x-1)^3}\cdot\frac{4}{\cancel{(x-1)^3}} = 0\cdot(x-1)^3 \longrightarrow 4=0 \;!$$

So:

f'': $-$ on the left of 1, $+$ on the right of 1

$$f''(0) = \frac{4}{(0-1)^3} = - \qquad f''(3) = \frac{4}{(3-1)^3} = +$$

Concave down on $(-\infty,1)$, concave up on $(1,\infty)$